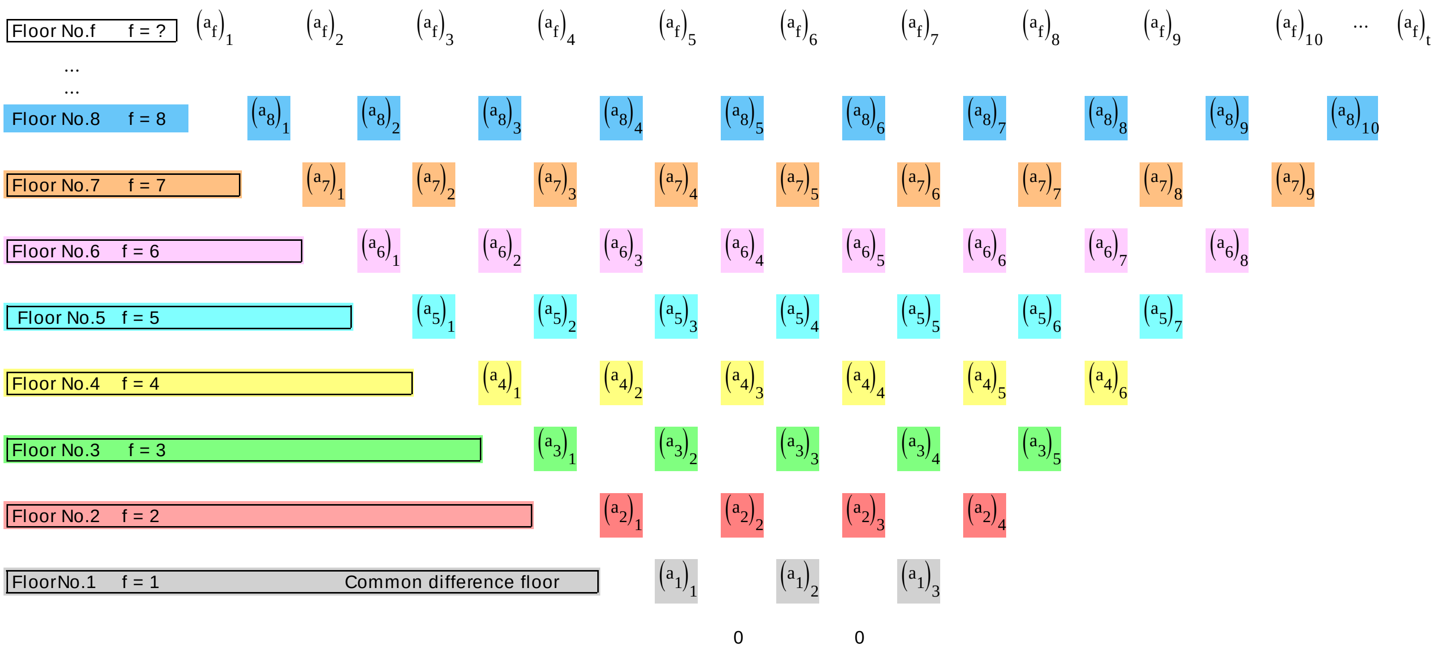
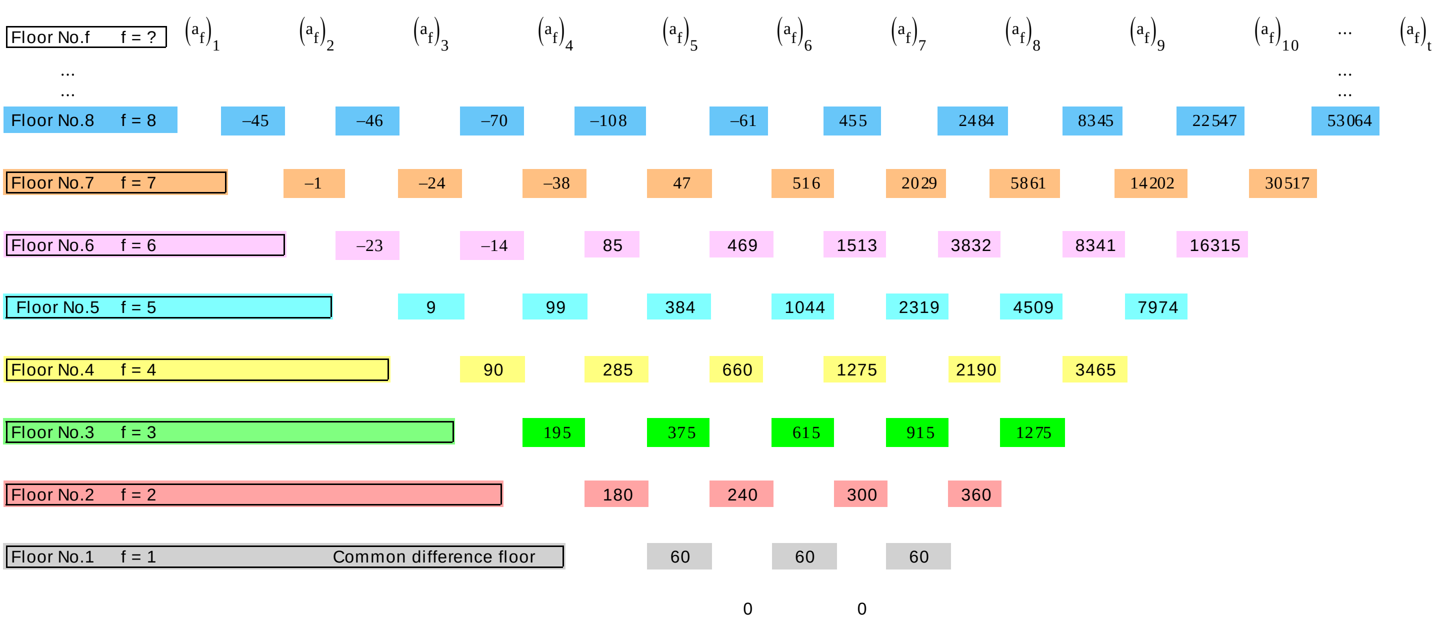
General formula for finding Sums of the Series in Arithmetic Sequences, basis on “***signed Stirling numbers of the first kind***” (Nth difference sequences), by “*Mohammad Reza Serajian Asl*”

In this article we are going to introduce the General formulas for finding the Sums of Series in all kinds of Arithmetic Sequences, (Nth difference sequences). As we know finding Sums of the Series in sequences is very hard and complicate; and it needs for a long times of calculating. But the introduced formulas in this Article make it too easy to do.

First of all, the (**f**th differences sequences) is as below figure.



And the numerical sample for above figure is as down.



In above figures, each one of the sequences makes a floor of building; and the difference of the terms on it, makes the lower sequence in lower floor of it.

In this article we are going to introduce a formula for summation of the series in above kinds of arithmetic sequences, basis on coefficients of the "***signed Stirling numbers of the first kind***".

In mentioned formulas, in $(a_f)_t$ "a", means arithmetic sequence, and the value of "f" is number of the floor which the arithmetic sequences are located on it, and the value of "t" is the number of the first "t" terms for summation.

Attention: for the reason that, there is similarity in signs of elements in ***sequence*** and ***Stirling numbers*****.** for preventing of interference between subjects, we put some changes in signs of elements in **sequence`s** subject. as "a" for sequence, and "f" means number of floor, and "t" means time of term in sequence.

For example, $(a_5)_7$ it means the seventh 7th term of the arithmetic sequence No.5 which located in floor No.5 or (f= 5 & t=7) and $\sum_{t=1}^{7}(a_5)_7$ it means the summation of the first 7 terms of the arithmetic Series. $\sum_{t=1}^{7}(a_5)_7 = 9 + 99 + 384 + 1044 + 2319 + 4509 + 7974 = 1633$

The formula for generating the formulas for summation of the first "t" terms of the series of arithmetic sequences.

$$\sum_{t=1}^{t}(a_f)_t = \left[(a_1)_1\cdot\frac{\sum\limits_{k=1,\,f=1}^{n,\,k=(f-0)}\left[s(n,k)\cdot t^{(f-0)}\right]}{(f-0)!}\right] + \left[(a_2)_1\cdot\frac{\sum\limits_{k=1,\,f=2}^{n,\,k=(f-1)}\left[s(n,k)\cdot t^{(f-1)}\right]}{(f-1)!}\right] + \left[(a_3)_1\cdot\frac{\sum\limits_{k=1,\,f=3}^{n,\,k=(f-2)}\left[s(n,k)\cdot t^{(f-2)}\right]}{(f-2)!}\right] + \ldots + \left[(a_f)_1\cdot\frac{\sum\limits_{k=1,\,f=f}^{n,\,k=[f-(f-1)]}\left[s(n,k)\cdot t^{[f-(f-1)]}\right]}{[f-(f-1)]!}\right]$$

Corrected form of above formula by using of double summation "double sigma" form is as below formula.

$$\sum_{t=1}^{t}(a_f)_t = \left[(a_1)_1\cdot\frac{\sum\limits_{k=1}^{n,\,k=(f-0)}\sum\limits_{f=1}^{(f-0)}\left[s(n,k)\cdot t^{(f-0)}\right]}{(f-0)!}\right] + \left[(a_2)_1\cdot\frac{\sum\limits_{k=1}^{n,\,k=(f-1)}\sum\limits_{f=2}^{(f-1)}\left[s(n,k)\cdot t^{(f-1)}\right]}{(f-1)!}\right] + \left[(a_3)_1\cdot\frac{\sum\limits_{k=1}^{n,\,k=(f-2)}\sum\limits_{f=3}^{(f-2)}\left[s(n,k)\cdot t^{(f-2)}\right]}{(f-2)!}\right] + \ldots + \left[(a_f)_1\cdot\frac{\sum\limits_{k=1}^{n,\,k=[f-(f-1)]}\sum\limits_{f=f}^{[f-(f-1)]}\left[s(n,k)\cdot t^{[f-(f-1)]}\right]}{[f-(f-1)]!}\right]$$

What is the meaning of the above main formula?

It means that, the first “*fractional expression*” is an equation in term of “t”, power or degree of “(f-0)”, and the parameters are a set of “*signed Stirling numbers of the first kind*” located in row No. “*equal with* (f-0)”.

For example the first “*fractional expression*” in above main formula for sequence $(a_7)_t$ is,

$$\sum_{t=1}^{t} (a_f)_t = \left[(a_1)_1 \cdot \frac{\sum_{k=1,\, f=1}^{n,\, k=(f-0)} \left[s(n,k) \cdot t^{(f-0)} \right]}{(f-0)!} \right] + \blacksquare \quad \blacksquare \rightarrow \quad \sum_{t=1}^{t} (a_7)_t = \left[(a_1)_1 \cdot \frac{\sum_{k=1,\, f=1}^{n,\, k=(7-0)} \left[s(7, k_{1,\ldots,7}) \cdot t^{(7-0)} \right]}{(7-0)!} \right] + \blacksquare$$

Signed Stirling numbers of the first kind:

n \ k	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8	k_9
n_1	1								
n_2	−1	1							
n_3	2	−3	1						
n_4	−6	11	−6	1					
n_5	24	−50	35	−10	1				
n_6	−120	274	−225	85	−15	1			
n_7	720	−1764	1624	−735	175	−21	1		
n_8	−5040	13068	−13132	6769	−1960	322	−28	1	
n_9	40320	−109584	118124	−67284	22449	−4536	546	−36	1

The first “*fractional expression*” in above “*main formula”* for sequence $(a_7)_t$ as sigma`s saying

$$\sum_{t=\blacksquare}^{t} (a_f)_t = (a_1)_1 \cdot \left[\frac{s(n_7,k_1) \cdot t^{(f_1-0)} + s(n_7,k_2)\, t^{(f_2-0)} + s(n_7,k_3)\, t^{(f_3-0)} + s(n_7,k_4)\, t^{(f_4-0)} + s(n_7,k_5)\, t^{(f_5-0)} + s(n_7,k_6)\, t^{(f_6-0)} + s(n_7,k_7)\, t^{(f_7-0)}}{(f-0)!} \right] + \blacksquare$$

Then after putting the related “*Stirling numbers*” as parameters of equation the first “*fractional expression*” will be create for $(a_7)_t$ sequence.

$$\sum_{t=\blacksquare}^{t} (a_7)_t = (a_1)_1 \cdot \left(\frac{720t^1 - 1764t^2 + 1624t^3 - 735t^4 + 175t^5 - 21 \cdot t^6 + 1 \cdot t^7}{7!} \right) + \blacksquare$$

The second "*fractional expression*" in above "*main formula*" for sequence $(a_7)_t$ as sigma`s saying

$$\sum_{t=\blacksquare}^{t}(a_7)_t=(a_1)_1\cdot(\text{First fraction})+(a_2)_1\cdot\left[\frac{s(n_6,k_1)\cdot t^{(f_1-1)}+s(n_6,k_2)t^{(f_2-1)}+s(n_6,k_3)t^{(f_3-1)}+s(n_6,k_4)t^{(f_4-1)}+s(n_6,k_5)t^{(f_5-1)}+s(n_6,k_6)t^{(f_6-1)}}{(f-1)!}\right]$$

Then after putting the related "*Stirling numbers*" as parameters of equation, the second "*fractional expression*" will be create for $(a_7)_t$ sequence

$$\sum_{t=\blacksquare}^{t}(a_7)_t=(a_1)_1\cdot(\text{First fraction})+(a_2)_1\cdot\left[\frac{-120t^{(2-1)}+274t^{(3-1)}-225t^{(4-1)}+85t^{(5-1)}-15t^{(6-1)}+1\cdot t^{(7-1)}}{(7-1)!}\right]$$

$$\left[\sum_{t=\blacksquare}^{t}(a_7)_t=(a_1)_1\cdot(\text{First fraction})+(a_2)_1\cdot\left(\frac{-120t+274t^2-225t^3+85t^4-15t^5+1\cdot t^6}{6!}\right)\right]+\blacksquare$$

The third "*fractional expression*" in above "*main formula*" for sequence $(a_7)_t$ as sigma`s saying

$$\sum_{t=\blacksquare}^{t}(a_7)_t=(a_1)_1\cdot(\text{First fraction})+(a_2)_1\cdot(\text{Second fraction})+(a_3)_1\cdot\left[\frac{s(n_5,k_1)\cdot t^{(f_1-2)}+s(n_5,k_2)t^{(f_2-2)}+s(n_5,k_3)t^{(f_3-2)}+s(n_5,k_4)t^{(f_4-2)}+s(n_5,k_5)t^{(f_5-2)}}{(f-2)!}\right]+\blacksquare$$

Then after putting the related "*Stirling numbers*" as parameters of equation the third "*fractional expression*" will be create for $(a_7)_t$ sequence

$$\sum_{t=\blacksquare}^{t}(a_7)_t=(a_1)_1\cdot(\text{First fraction})+(a_2)_1\cdot(\text{Second fraction})+(a_3)_1\cdot\left[\frac{24t^{(f_1-2)}-50t^{(f_2-2)}+35t^{(f_3-2)}-10t^{(f_4-2)}+1\cdot t^{(f_5-2)}}{(f-2)!}\right]+\ldots+\blacksquare$$

And then

$$\sum_{t=\blacksquare}^{t}(a_7)_t=(a_1)_1\cdot(\text{First fraction})+(a_2)_1\cdot(\text{Second fraction})+(a_3)_1\cdot\left(\frac{24t-50t^2+35t^3-10t^4+t^5}{5!}\right)+\ldots+\blacksquare$$

Now we are completed the third "*fractional expression*" and in continue by above mentioned manner, (by reducing or increasing the lower and upper limits of sigma`s parameters, as value of "f, n, k" we should create the other "*fractional expression*" one by one till the last one of them.

The last “*fractional expression*” in above “*main formula”* for sequence $\left(a_7\right)_t$ as sigma`s saying

$$\sum_{t=}^{t}\left(a_7\right)_t = \left(a_1\right)_1\cdot(\text{First fraction}) + \left(a_2\right)_1\cdot(\text{Second fraction}) + \left(a_3\right)_1\cdot(\text{Third fraction}) + \ldots + \left(a_f\right)_1\cdot\frac{\sum_{k=1,\,f=f}^{n,\,k=[f-(f-1)]}\left[s(n,k)\cdot t^{[f-(f-1)]}\right]}{[f-(f-1)]!}$$

Then after putting the related “*Stirling numbers*” as parameters of equation the last, 7th (for present example) *“fractional expression”* will be create for $\left(a_7\right)_t$ sequence

$$\sum_{t=}^{t}\left(a_7\right)_t = \left(a_1\right)_1\cdot(\text{First fraction}) + \left(a_2\right)_1\cdot(\text{Second fraction}) + \left(a_3\right)_1\cdot(\text{Third fraction}) + \ldots + \left(a_7\right)_1\cdot\frac{\sum_{k=1,\,f=7}^{n,\,k=[7-(7-1)]}\left[s(1,1)\cdot t^{[7-(7-1)]}\right]}{[7-(7-1)]!}$$

$$\sum_{t=}^{t}\left(a_7\right)_t = \left(a_1\right)_1\cdot(\text{1th fraction}) + \left(a_2\right)_1\cdot(\text{2th fraction}) + \left(a_3\right)_1\cdot(\text{3th fraction}) + \left(a_4\right)_1\cdot(\text{4th fraction}) + \left(a_5\right)_1\cdot(\text{5th fraction}) + \left(a_6\right)_1\cdot(\text{6th fraction}) + \left(a_7\right)_1\cdot\frac{1\cdot t^1}{1!}$$

The general fourmula for summation of the first “t” terms of the series of arithmetic sequences $\left(a_7\right)_t$ locates in floor No.7 with a sample for sequence $\left(a_7\right)_8$

$$\sum_{t=1}^{t}\left(a_7\right)_t = \left(a_1\right)_1\cdot\left(\frac{720t - 1764t^2 + 1624t^3 - 735t^4 + 175t^5 - 21\cdot t^6 + t^7}{7!}\right) + \left(a_2\right)_1\cdot\left(\frac{-120t + 274t^2 - 225t^3 + 85t^4 - 15t^5 + t^6}{6!}\right) + \left(a_3\right)_1\cdot\left(\frac{24t - 50t^2 + 35t^3 - 10t^4 + t^5}{5!}\right) + \left(a_4\right)_1\cdot\left(\frac{-6\cdot t + 11\cdot t^2 - 6\cdot t^3 + t^4}{4!}\right) + \left(a_5\right)_1\cdot\left(\frac{2\cdot t - 3\cdot t^2 + t^3}{3!}\right) + \left(a_6\right)_1\cdot\left(\frac{-t + t^2}{2!}\right) + \left(a_7\right)_1\cdot\left(\frac{t}{1!}\right) =$$

$$-1 + -24 + -38 + 47 + 516 + 2029 + 5861 + 14202 = 22592$$

$$60\left(\frac{720\cdot 8 - 1764\cdot 8^2 + 1624\cdot 8^3 - 735\cdot 8^4 + 175\cdot 8^5 - 21\cdot 8^6 + 8^7}{7!}\right) + 180\left(\frac{-120\cdot 8 + 274\cdot 8^2 - 225\cdot 8^3 + 85\cdot 8^4 - 15\cdot 8^5 + 8^6}{6!}\right) + 195\left(\frac{24\cdot 8 - 50\cdot 8^2 + 35\cdot 8^3 - 10\cdot 8^4 + 8^5}{5!}\right) + 90\cdot\left(\frac{-6\cdot 8 + 11\cdot 8^2 - 6\cdot 8^3 + 8^4}{4!}\right) + 9\cdot\left(\frac{2\cdot 8 - 3\cdot 8^2 + 8^3}{3!}\right) + (-23)\cdot\left(\frac{-8 + 8^2}{2!}\right) + (-1)\cdot\left(\frac{8}{1!}\right) = 22592$$

By the main formula we can create the formulas for other sequences located in lower and upper floors

For example: the formula for the sequence $\left(a_8\right)_t$ and an example for $\left(a_8\right)_{10}$ are as below.

$$\sum_{t=1}^{t}\left(a_8\right)_t = \left(a_1\right)_1\cdot\left(\frac{-5040t + 13068t^2 - 13132t^3 + 6769t^4 - 1960t^5 + 322t^6 - 28\cdot t^7 + t^8}{8!}\right) + \left(a_2\right)_1\cdot\left(\frac{720t - 1764t^2 + 1624t^3 - 735t^4 + 175t^5 - 21\cdot t^6 + t^7}{7!}\right) + \left(a_3\right)_1\cdot\left(\frac{-120t + 274t^2 - 225t^3 + 85t^4 - 15t^5 + t^6}{6!}\right) + \left(a_4\right)_1\cdot\left(\frac{24t - 50t^2 + 35t^3 - 10t^4 + t^5}{5!}\right) + \left(a_5\right)_1\cdot\left(\frac{-6\cdot t + 11\cdot t^2 - 6\cdot t^3 + t^4}{4!}\right) + \left(a_6\right)_1\cdot\left(\frac{2\cdot t - 3\cdot t^2 + t^3}{3!}\right) + \left(a_7\right)_1\cdot\left(\frac{-t + t^2}{2!}\right) + \left(a_8\right)_1\cdot\left(\frac{t}{1!}\right) =$$

$$(-45) + (-46) + (-70) + (-108) + (-61) + (455) + (2484) + (8345) + (22547) + (53064) = 86565$$

$$60\left(\frac{10^8 - 28\times 10^7 + 322\times 10^6 - 1960\times 10^5 + 6769\times 10^4 - 13132\times 10^3 + 13068\times 10^2 - 5040\times 10}{8!}\right) + 180\left(\frac{10^7 - 21\times 10^6 + 175\times 10^5 - 735\times 10^4 + 1624\times 10^3 - 1764\times 10^2 + 720\times 10}{7!}\right) + 195\left(\frac{10^6 - 15\times 10^5 + 85\times 10^4 - 225\times 10^3 + 274\,10^2 - 120\times 10}{6!}\right) + 90\left(\frac{10^5 - 10\times 10^4 + 35\times 10^3 - 50\times 10^2 + 24\cdot 10}{5!}\right) + 9\left(\frac{10^4 - 6\times 10^3 + 11\times 10^2 - 6\times 10}{4!}\right) + -23\left(\frac{10^3 - 3\times 10^2 + 2\times 10}{3!}\right) + -1\cdot\left(\frac{10^2 - 10}{2!}\right) + -45\left(\frac{10}{1!}\right) = 86565$$

For example: the formula for the sequence $\left(a_9\right)_t$ and an example for $\left(a_9\right)_8$ are as below.

$$\sum_{t=1}^{8}(a_9)_8=\left[(a_1)_1\cdot\frac{\left[\sum_{f=1,k=1}^{n,k,(f-0)=(9-0)}\left[s(9,k)\cdot 8^{(9-0)}\right]\right]}{(9-0)!}\right]+\left[\left[a_{(2)}\right]_1\cdot\frac{\left[\sum_{f=2,k=1}^{n,k,(f-1)=(9-1)}\left[s(8,k)\cdot 8^{(9-1)}\right]\right]}{(9-1)!}\right]+\left[\left[a_{(3)}\right]_1\cdot\frac{\left[\sum_{f=3,k=1}^{n,k,(f-2)=(9-2)}\left[s(7,k)\cdot 8^{(9-2)}\right]\right]}{(9-2)!}\right]+\ldots+\left[(a_9)_1\cdot\frac{\left[\sum_{f=9,k=1}^{n,k,[f-(f-1)]=[9-(9-1)]}\left[s(1,k)\cdot 8^{[9-(9-1)]}\right]\right]}{[9-(9-1)]!}\right]$$

$$60\cdot\left(\frac{40320\cdot 8-109584\cdot 8^2+118124\cdot 8^3-67284\cdot 8^4+22449\cdot 8^5-4536\cdot 8^6+546\cdot 8^7-36\cdot 8^8+8^9}{9!}\right)+180\cdot\left(\frac{-5040\cdot 8+13068\cdot 8^2-13132\cdot 8^3+6769\cdot 8^4-1960\cdot 8^5+322\cdot 8^6-28\cdot 8^7+8^8}{8!}\right)+195\cdot\left(\frac{720\cdot 8-1764\cdot 8^2+1624\cdot 8^3-735\cdot 8^4+175\cdot 8^5-21\cdot 8^6+8^7}{7!}\right)+90\cdot\left(\frac{-120\cdot 8+274\cdot 8^2-225\cdot 8^3+85\cdot 8^4-15\cdot 8^5+8^6}{6!}\right)+9\cdot\left(\frac{24\cdot 8-50\cdot 8^2+35\cdot 8^3-10\cdot 8^4+8^5}{5!}\right)+(-23)\cdot\left(\frac{-6\cdot 8+11\cdot 8^2-6\cdot 8^3+8^4}{4!}\right)+(-1)\cdot\left(\frac{2\cdot 8-3\cdot 8^2+8^3}{3!}\right)+(-45)\cdot\left(\frac{-8+8^2}{2!}\right)+(-0.2)\cdot\left(\frac{8}{1!}\right)=1836.4$$

Important point: by adding the value of the first term of upper Sequence to formula of lower sequence we can determine the terms of upper sequences.

Mohammad Reza Serajian Asl

Key words: sequences, series, arithmetic sequences, Stirling numbers of the first kind,
Nth difference sequences,

1- Wikipedia “Stirling numbers of the first kind”
2- OEIS.