General recurrence relation formula for determin all kinds of the Stirling and r-Stirling numbers, basis on prismatic numerical arrays

The r-Stirling numbers of the fourth kind in the case, r = 2.

1, 8, 1, 80, 20, 1, 960, 360, 36, 1, 13440, 6720, 1008, 56, 1, 215040, 134400, 26880, 2240, 80, 1, 3870720, 2903040, 725760, 80640, 4320, 108, 1, 77414400, 67737600, 20321280, 2822400, 201600, 7560, 140, 1, 1703116800, 1703116800, 596090880, 99348480, 8870400, 443520, 12320, 176, 1, 40874803200, 45984153600, 18393661440, 3576545280, 383201280, 23950080, 887040, 19008, 216, 1

These sequences are the r-Stirling numbers of the fourth kind in the case (r = 2) basis on a prismatic numerical array consisted the slices of triangular arrays, which is defined by:

Stx(n+1,k,r) = Stx(n,k,r)+ [Stx(n,k-1,r)]*[(2^(x-3)]*[(-k+ 2*(n+r)]

In above recurrence relation the values of (x) indicate the kinds of Stirling numbers, for example the number (4) in (St4), indicates Stirling numbers of the fourth kind, which it is similar to A079621.

The values of (n), (k), (r), respectively indicate the numbers of (rows), (pages or arrays), (columns), in a prismatic numerical array.

The Stirling and r-Stirling numbers of the first and second kinds can be generate by prismatic arrays basis on recurrence relations as below.

1st kind: St1(n+1,k,r) = St1(n,k,r) + St1(n,k-1,r)*[(n+r-1)].

2nd kind: St2(n+1,k,r) = St2(n,k,r) + St2(n,k-1,r)*[(-k)+(n+r+1)].

The Stirling and r-Stirling numbers bigger than second kind x > 2 can be generate by prismatic arrays of numbers basis on recurrence relations as below formula.

Stx(n+1,k,r) = Stx(n,k,r)+[Stx(n,k-1,r)]*[(2^(x-3)]*[(-k+ 2*(n+r)].

The cases of (r > 1), in recurrence relations formulas give the related r-Stirling numbers.

The prismatic arrays basis on recurrence relations are formed of consecutive square arrays as successive pages, that all values of the first array or page (k = 1), are one (1), and the first row of it, is number one (n = 1), the values of the second square array or page (k = 2) can be calculate by using of the recurrence relations formulas. The first row in second array is (n = 2), and so on to the end for next other arrays or pages, in this case it will be a 3D prismatic array which the successively and respectively located numbers in a same row (n), and column (r), in consecutive arrays (k), make sets of Stirling or r-Stirling numbers of the x-th kinds.

As there is evident, the difference between the relations is, only the–, coefficients of the relations, which they are sum of the prior coefficients– it means that the coefficients of the relations in first kind, plus the coefficients of the relations in second kind, give the coefficients of the relations in third kind or Lah numbers, and sum of the first kind and second kind plus third kind of coefficients, give the coefficients of the relation for fourth kind and etc; as below relations.

(coeff.of 1st kind)+(coeff.of 2nd kind)+(coeff.of 3rd kind)==➔(coeff.of fourth kind).

In general case, the coefficient of x-th kind as below relation.

(coeff. of 1st kind)+(coeff. of 2nd kind)+(coeff. of 3rd kind)+(coeff. of 4th kind) +...+ (coeff. of (x-1)-th kind) ==➔ (coeff. of x-th kind)

For example: the coefficient of the recurrence relation for the Stirling and r-Stirling numbers of the fourth kind is as below relation.

[coeff.of 1st kind]+[coeff.of 2nd kind]+[coeff.of 3rd kind] = [coeff.of 4th kind]
[(n+r-1)]+[(-k)+(n+r+1)]+[(-k)+ 2*(n+r)]+[(-2*k)+4*(n+r)] = [(-4*k)+8*(n+r)]

Recurrence relations for Stirling and r-Stirling numbers of the (1st, 2th, 3th,... , x-th)

```
1st kind: St1(n+1,k,r) = St1(n,k,r) + St1(n,k-1,r)*[(n+r-1)];
2nd kind: St2(n+1,k,r) = St2(n,k,r) + St2(n,k-1,r)*[(-k)+(n+r+1)];
3rd kind: St3(n+1,k,r) = St3(n,k,r) + St3(n,k-1,r)*[(-k)+ 2*(n+r)];
4th kind: St4(n+1,k,r) = St4(n,k,r) + St4(n,k-1,r)*[(-2*k)+4*(n+r)];
5th kind: St5(n+1,k,r) = St5(n,k,r) + St5(n,k-1,r)*[(-4*k)+8*(n+r)];
6th kind: St6(n+1,k,r) = St6(n,k,r) + St6(n,k-1,r)*[(-8*k)+16*(n+r)];
7th kind: St7(n+1,k,r) = St7(n,k,r) + St7(n,k-1,r)*[(-16*k)+32*(n+r)];
```

The general recurrence relations for Stirling and r-Stirling numbers; greater than second kinds (x > 2) is as below formula.
x-th kind: Stx(n+1,k,r) = Stx(n,k,r)+[Stx(n,k-1,r)]*{[-2^(x-3)]*k +[2^(x-2)]*(n+r)]}

Stirling and r-Stirling numbers smaller than third kinds(x < 3)
St1(n+1,k,r) = St1(n,k,r) + St1(n,k-1,r)*[(n+r-1)]
St2(n+1,k,r) = St2(n,k,r) + St2(n,k-1,r)*[(-k)+(n+r+1)]
Stirling and r-Stirling numbers; greater than second kinds(x > 2)
Stx(n+1,k,r) = Stx(n,k,r)+ [Stx(n,k-1,r)]*[(2^(x-3)]*[(-k+ 2*(n+r)]
Prismatic arrays for Stirling and r-Stirling numbers in other hand are formed of square arrays basis on related recurrence relations.
The numbers located in same row (n), and same column (r), and respective and consecutive of arrays or pages (k), make a set of Stirling or r-Stirling numbers of the fourth kind.
For example: The set of numbers locate in (n = 5), and,
(k = 1,2,3,...,5), and (r = 2), is (1, 56, 1008, 6720, 13440) which is a set of r-Stirling numbers of the fourth kind

```
****************************************
n\r|........1........2........3........4
======================================== k = 1
1..|........1........1........1........1
2..|........1........1........1........1
3..|........1........1........1........1
4..|........1........1........1........1
5..|........1........1........1........1
6..|........1........1........1........1
****************************************
n\r|........1........2........3........4
======================================== k = 2
2..|........4........8.......12.......16
3..|.......12.......20.......28.......36
4..|.......24.......36.......48.......60
5..|.......40.......56.......72.......88
6..|.......60.......80......100......120
****************************************
n\r|........1........2........3........4
======================================== k = 3
3..|.......24.......80......168......288
4..|......144......360......672.....1080
5..|......480.....1008.....1728.....2640
```

```
6..|.....1200.....2240.....3600.....5280
****************************************
n\r|........1........2........3........4
======================================== k = 4
4..|......192......960.....2688.....5760
.|.....1920.....6720....16128....31680
6..|.....9600....26880....57600...105600
****************************************
n\r|........1........2........3........4
======================================== k = 5
5..|.....1920....13440....48384...126720
6..|....28800...134400...403200...950400
****************************************
n\r|........1........2........3........4
======================================== k = 6
6..|....23040...215040...967680..3041280
****************************************.
```

A prismatic numerical array, consists the slices of triangular arrays.
For example: the above mentioned prismatic array for Stirling and r-Stirling numbers of the fourth kind, consists of triangular arrays as down figures.
In the case (r = 1), the matrix square unsigned Lah triangle A079621.

The recurrence relations formula for triangular forms of array is as below.

Stx(n+1,(n-t+2),r) = Stx(n,(n-t+1),r)+[Stx(n,(n-t+2),r)]*[2^(x-3)]*(t-n)+[2^(x-2)]*(n+r-1)].

The recurrence relations formula for triangular forms of, the Stirling numbers of the forth kind is as below.

St4(n+1,(n-t+2),1) = St4(n,(n-t+1),1)+[Stx(n,(n-t+2),1)]*[2^(4-3)]*(t-n)+[2^(4-2)]*(n+1-1)].

The recurrence relations formula for triangular forms of the r-Stirling numbers of the forth kind in the case (r = 2) is as below.

St4(n+1,(n-t+2),2) = St4(n,(n-t+1),2)+[Stx(n,(n-t+2),2)]*[2^(4-3)]*(t-n)+[2^(4-2)]*(n+2-1)].

Triangular array of the Stirling and r-Stirling numbers of the forth kind.
(k = n-t+2)in prismatic array and (t = n-k+2) in triangular array.

```
n\t|......1.......2......3......4....5...6
  ==========================================
1..|      1;
2..|      4,      1;
3..|     24,     12,     1;
4..|    192,    144,    24,     1;
5..|   1920,   1920,   480,    40,   1;
6..|  23040,  28800,  9600,  1200,  60,   1;
.........................................
.......................................
```

In the case (r = 2), the r-Stirling numbers of the fourth kind.

```
n\t|......1.......2......3......4....5...6
  ==========================================
1..|      1;
2..|      8,      1;
3..|     80,     20,     1;
4..|    960,    360,    36,     1;
5..|  13440,   6720,  1008,    56,   1;
6..| 215040, 134400, 26880,  2240,  80,  1;
.........................................
.........................................
```

And the other cases of (r), for r-Stirling numbers of the fourth kind.

Key words: Stirling and r-Stirling numbers of the x-th kinds; Stirling numbers of the fourth kind; Stirling numbers of the all kinds; prismatic numerical arrays for Stirling numbers.

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